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Journal of Geometry and Physics 19 (1996) 123–142
JOURNAL OF GEOMETRY AND PHYSICS

Composite A,B,C,D-IRF-model invariants

Olaf J. Backofen [1]

Institut für Theoretische Physik, Universität Saarbrücken, FR. 10.1, D-66041 Saarbrücken, Germany

Received 25 January 1995; revised 11 April 1995

Abstract

In this work the introduction of generalized A,B,C,D interaction-round-a-face model invariants related to composite braid group representations will be proposed. The invariant polynomials are obtained in the framework of Witten's Chern–Simons theory summarizing recent works on link invariants. The primary intention is to present explicitly neglected results in the latter area and to outline in a pedagogical way the computation of a variety of known and new invariants. The close relationship of the topological interpretation of link invariants and the notion of generalized knot polynomials derived from integrable models in statistical mechanics is emphasized.

Keywords: Composite knot invariants; Chern–Simons theory; Lie algebras; Duality
1991 MSC: 57M25, 57N10, 20F36, 82B23

1. Introduction

In recent years several different approaches in knot theory have emerged. Connections to a variety of problems in mathematics and mathematical physics appeared as a consequence [1].

In order to put the present work in an appropriate context, a brief review of the development of modern knot theory will be given. A first relationship between polynomial invariants of links and statistical mechanics was already implicitly contained in the pioneering paper of Jones [2], where Jones introduced his famous polynomial via a study of certain finite-dimensional von Neumann algebras. This implicit relationship was made explicit for the first time by Kauffman [3,4] computing a partition function from the knot or link diagrams. Soon after generalizations were developed, especially the HOMFLY [5] and Kauffman polynomial [6], giving rise to more powerful invariants.

[1] E-mail: ph11ahob@sbusol.rz.uni-sb.de.

SSDI 0393-0440(95)00027-5

The discovery of a new intimate connection between knot theory and statistical mechanics was established independently by Jones [7], Wadati et al. [8] and Turaev [9], yielding invariants with skein relations of higher order related to solutions of the Yang–Baxter equation. Considering exactly solvable models with Lie algebra symmetry of Cartan's type A_{n-1}, B_n, C_n and D_n, the A,B,C,D interaction-round-a-face (IRF) model invariants [8] were introduced. Similarly hierarchies of N-vertex model invariants and their two-variable generalizations corresponding to a composite braid group representation were developed. Both the A-IRF-model invariant and the extended two-vertex-model invariant represent a HOMFLY polynomial.

The consideration of state sum models related to the fundamental representation of $SU(n)$ by Kauffman [10] engendered a new valuable viewpoint of link invariants. For the more complicated examples of higher-dimensional representations, the extension was made by introducing the annihilation diagrams for the type of Akutsu–Wadati's polynomial calculations [11–14] emphasizing once more the close relationship between the scattering picture, the Yang–Baxter equation and the associated state sum models. Furthermore Li and Ge calculated link invariants for some non-standard representations of the braid group [15,16].

A remarkable renaissance in the interaction of knot theory and mathematical physics originated in obtaining invariant polynomials in the framework of Witten's Chern–Simons topological gauge theory. In most of the previous work in knot theory the evaluation of the invariants is based on two-dimensional (2D) projections of links, or algebraic approaches. However, knots are living intrinsically in three dimensions and so a three-dimensional definition was desirable. It was Witten who provided the answer in his celebrated paper [17] by describing knot polynomials as vacuum expectation values of Wilson line operators in a (2+1)D quantum field theory based on the pure Chern–Simons (CS) action. Topological surgery then allowed a study of knot invariants in an arbitrary three-manifold. Besides its mathematical advantages CS theory also provides a unifying 3D viewpoint for (1+1)D conformal field theory [17–19] as well as (2+1)D quantum gravity [20].

Witten's derivation of link invariants considers monodromy operations of braid matrices related to the fundamental representations of Lie groups of Cartan's type A_{n-1}. Using the Lie group $SU(n)$, the HOMFLY ($n \in \mathbb{N}$) and Jones Polynomial ($n = 2$) are obtained, respectively. As a consequence generalizations of Witten's theory were proposed successively. Applying the theory to the classical Lie groups of Cartan's classification A_{n-1}, B_n, C_n and D_n in fundamental representation, generalized skein relations were deduced by similar methods [21–26], implying an inherent 3D introduction of the A,B,C,D-IRF-model invariants of Akutsu et al. [22–24]. Kauffman polynomials correspond to $SO(n)$ Chern–Simons theory. The compact simple Lie algebras $E_{6,7,8}$, F_4 and G_2 were examined by Hayashi [26] and Ge et al. [22]. Furthermore the theory was extended to the supersymmetric Lie groups $SU(n|n')$ and $OSp(n|2n')$ in fundamental representation [25] and the non-standard invariants of Li and Ge [24].

The generalization of Witten's Chern–Simons theory to higher-dimensional representations such as the spin-s, i.e. the composite representation, especially for $SU(2)$, generated a new useful viewpoint of knot theory suggesting the possibility to define the

$N = 2s + 1$ vertex-model invariants by quantum field theoretical means [22–24,27,28]. Following Govindarajan et al. the method can be persued to multicolored links where different representations of $SU(2)$ [29] or $SU(n)$ [30] are placed on the component knots. Unfortunately for multicolored links the corresponding generalized Alexander–Conway relations do not admit recursive solutions in general and new direct methods to obtain the invariants had to be developed [29]. The general properties of universal link polynomials for a generic real simple Lie algebra were examined in detail by Guadagnini [31]. Recently Guadagnini and Pilo [32,33] achieved to explore far-reaching consequences for $SU(3)$ CS theory using in particular composite Wilson line operators of multicolored links. Isidro et al. [27] and Govindarajan [34] have also studied invariants for toral knots from minimal conformal models.

The present approach proposes to extend Witten's interpretation of link invariants to higher-dimensional composite representations of arbitrary compact semi-simple Lie groups. In the same manner as the composite braid group analysis generalizes the N-vertex polynomials using Yang–Baxter state models, topological field theory may engender generalized or composite A,B,C,D-IRF-model invariants. Hereby the topological derivation is of particular importance since the well-known fusion method to obtain composite braid group operators of Wadati et al. [8,11] is applicable only for vertex or IRF models with quadratic minimal polynomials such as the IRF models associated with $SU(n)$ [35]. This implies that the composite link polynomials of type B_n, C_n or D_n with cubic braid group reduction relations are new. The invariants depending on the gauge group G and spin-s will be calculated explicitly providing a possibility to classify the previously found vertex- and IRF-model invariants. While Witten's interpretation is a 3D quantum field theoretical (QFT) treatment of Jones' original approach related to a Markov trace, the present derivation establishes a 3D QFT version of invariants derived from generalized Ocneanu traces.

The paper is organized as follows: Section 2 will review the composite link polynomial construction of Wadati et al. [8,11], Witten CS theory [17] and the derivation of A,B,C,D-IRF-model invariants [21–26] will be summarized in Section 3. In Section 4 the topological approach will be extended to composite invariants. The paper concludes with an outlook to a corresponding quantum group interpretation in Section 5.

2. Composite link invariants

Wadati et al. [8,11] presented a general prescription to construct a representation of a braid group $\mathcal{B}_k$ from the Boltzmann weights of a solvable model satisfying the Yang–Baxter relation [8]. A sequence of solvable vertex- or IRF-models with quadratic minimal polynomials and $SU(n)$ symmetries generated new braid group representations. The choice of associated Markov traces enabled the construction of new hierarchies of link polynomials. Starting from the generators $\{g_i\}$ of the braid group $\mathcal{B}_k$ the composite braid group operators $\{G_i\}$ were obtained using symmetry projectors P. A composite string is formed by combining m strings and attaching a projector P at each end (Fig. 1). This process known as fusion procedure corresponds to the construction of higher spin S-matrices from lower spin

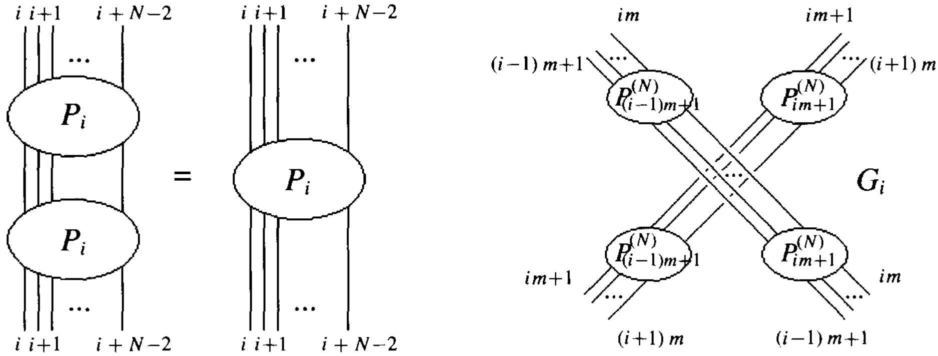


Fig. 1. The composite string and corresponding braid group representation.

S-matrices [36]. For example, from a pair of spin-$\frac{1}{2}$ particles one can form two composite particles with spin 0 and 1, respectively. The projector P_i is assigned to select the spin $s = \frac{1}{2}m$, where s is related to the state number N of the N-vertex model by

$$s = \tfrac{1}{2}(N-1) = \tfrac{1}{2}m. \tag{1}$$

Preparing k sets of m strings and introducing an operator $g_i^{(j)}$ (cf. [8]),

$$g_i^{(j)} = g_{im+1-j}g_{im+2-j}\cdots g_{(i+1)m-j} \quad (j = 1,\ldots,N-1), \tag{2}$$

the generators $\{G_i\}$ of the composite braid group $\mathcal{B}_k^{[s]}$ are defined by

$$G_i = P^{(N)}_{(i-1)m+1}P^{(N)}_{im+1}\,g_i^{(1)}g_i^{(2)}\cdots g_i^{(N-1)}\,P^{(N)}_{(i-1)m+1}P^{(N)}_{im+1}. \tag{3}$$

The projectors $P_i^{(N)}$ may be evaluated explicitly for solvable vertex- and IRF-models using algebraic calculations [8] or conformal field theory (cf. Section 4). The generators $G_1,\ldots,G_{k-1}$ satisfy the defining relation of the braid group. Consequently following Wadati et al. [8,11], a generalized Ocneanu trace was introduced:

$$\psi^{[s]}(A) = \frac{\psi(A)}{[\psi(P_j)]^k}, \quad A \in \mathcal{B}_k^{[s]}. \tag{4}$$

$\psi^{[s]}(\,\cdot\,)$ satisfies the Markov properties leading to the well-known Akutsu–Wadati composite (two-variable) invariants

$$\alpha_\omega^{[s]} = (\overline{Z}Z)^{-(k-1)/2}(\overline{Z}/Z)^{e(A)/2}\psi^{[s]}(A), \quad A \in \mathcal{B}_k^{[s]}, \tag{5}$$

with the abbreviations

$$\omega \equiv \frac{\psi(g_j^{-1})}{\psi(g_j)}, \qquad Z \equiv \psi^{[s]}(G_j), \qquad \overline{Z} \equiv \psi^{[s]}(G_j^{-1}). \tag{6}$$

$e(A)$ is as usually the exponent sum of the generators in $\mathcal{B}_k^{[s]}$.

In the case of the A_{n-1} N-vertex (and IRF) model the new two-variable polynomials were evaluated explicitly in [8] making use of

$$Z = \frac{(1-t)\cdots(1-t^{N-1})}{(1-\omega t)\cdots(1-\omega t^{N-1})}, \qquad \overline{Z} = \omega^{N-1} Z \quad (\omega = t^{n-1}). \tag{7}$$

This yields a HOMFLY type polynomial for $N = 2$,

$$\alpha(L_+) \;=\; \omega^{1/2}(1-t)\,\alpha(L_0) + \omega t\,\alpha(L_-), \tag{8}$$

and a hierarchy of new polynomials if N is considered as a continuous parameter. For $N \geq 3$ the invariants derived comply with a skein relation of higher order, e.g., for $N = 3$,

$$\alpha(L_{2+}) = \omega(1-t^2+t^3)\alpha(L_+) + \omega^2(t^2-t^3+t^5)\alpha(L_0) - \omega^3 t^5 \alpha(L_-) = 0. \tag{9}$$

The choice of the spin projector P_i is not unique implying the possibility to generalize the composite string representation. Every projector being an eigenvector of the full twist Δ_i^2 with eigenvalue α_λ,

$$P_i^\lambda \Delta_i^2 \;=\; \alpha_\lambda P_i^\lambda, \tag{10}$$

will ensure the Markov properties of the new trace function. The following generalization of the invariants is straightforward:

$$\alpha_\omega^{[\lambda]} \;=\; (\overline{Z}_\lambda Z_\lambda)^{-(k-1)/2}(\overline{Z}_\lambda/Z_\lambda)^{e(A)/2}\psi^{[\lambda]}(A), \quad A \in \mathcal{B}_k^{[\lambda]}. \tag{11}$$

Unfortunately the fusion method fails if the braid group generators $\{g_i\}$ satisfy cubic reduction relations as in the case of solvable models associated with special orthogonal or symplectic Lie groups implying that new methods have to be considered.

3. Invariants and Chern–Simons theory

The simplest topological field theory with the pure CS action is a 3D model for a vector field A with action

$$\mathcal{L}_{\mathrm{CS}} \;=\; \frac{k}{4\pi}\int Tr(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A). \tag{12}$$

The basic objects of Witten's topological knot theory are the vacuum expectation values on the three-sphere S^3 of Wilson line operators in Chern–Simons gauge theory with arbitrary compact semi-simple Lie group G. A link L in S^3 may be considered as a disjoint union of circles C_i oriented and labeled with a choice of representation R_i of G. Here the R_i are the elements of the finite set $\mathcal{R}$ of representations of G that are highest weight integrable representations of the loop group $\mathcal{L}G$ at level k. The (unnormalized) expectation value of L is given by Witten's partition function

$$\langle L \rangle = Z_{G,R}(M,L) = \int \mathcal{D}A\, \mathrm{e}^{\mathrm{i}\mathcal{L}_{\mathrm{CS}}}\, \mathcal{O}(A). \tag{13}$$

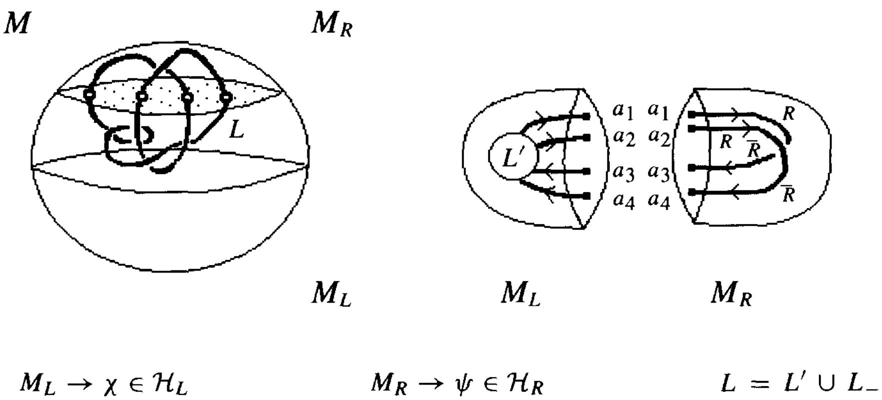


Fig. 2. The three-manifold M containing the link L is cut into a simple piece M_R and a complicated piece M_L.

The most basic gauge invariant functional $\mathcal{O}(A)$ to consider is the Wilson line operator

$$\mathcal{O}(A) \equiv \prod_i W_{R_i}(C_i) = \prod_i Tr_{R_i}\, P \exp \oint_{C_i} A, \tag{14}$$

where $P \exp \oint_{C_i} A$ denotes the holonomy around the component C_i and P is the path ordering operator.

Following Witten's approach $Z_{G,R}(M, L)$ can be evaluated employing an algorithm for untangling knots. The link L is embedded in a three-manifold M in such a way that one crossing remains in a part M_R and the rest of the link is located in the part M_L. The manifold M is the connected sum of the two pieces M_R and M_L joined along a two-sphere S^2 with four marked points $a_1, \ldots, a_4$ combined by two Wilson lines as illustrated in Fig. 2.

The link represented by Wilson lines with corresponding representations R and R' there exist physical Hilbert spaces $\mathcal{H}_R$ and $\mathcal{H}_L$ associated with the boundaries of M_R and M_L. According to Witten [17] the Hilbert spaces are of dimension N if the direct product of the irreducible representation R decomposes into N distinct irreducible representations (IR) of G:

$$R \otimes R = \bigoplus_{i=1}^{N} E_i. \tag{15}$$

Supposing that χ and ψ are the bases of the Hilbert spaces $\mathcal{H}_L$ and $\mathcal{H}_R$, respectively, then the partition function is obtained through the contraction

$$Z_{G,R}(M, L) = \langle \chi | \psi \rangle. \tag{16}$$

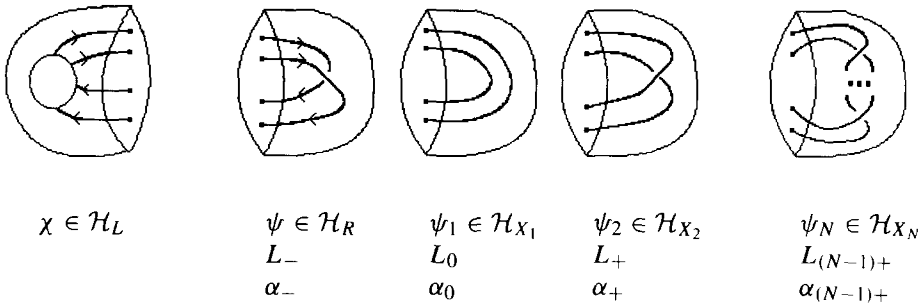


Fig. 3. In order to obtain skein relations the three-manifold M_R is replaced by the manifolds X_i ($i = 1, \ldots, N$).

The dimension of the Hilbert spaces being N, any $N+1$ vectors obey a relation of linear dependence of the form

$$\alpha\psi + \alpha_1\psi_1 + \cdots + \alpha_N\psi_N = 0. \tag{17}$$

The well-known method (cf. [17]) to get additional vectors in $\mathcal{H}$ is to replace M_R (in Fig. 2) by any other three-manifolds X_i ($i = 1, \ldots, N$) with same boundary and suitable strings in X_i and the Feynman integral will generate new vectors in $\mathcal{H}$. Choosing the string configurations of the usual crossing types $L_-, L_0, L_+, \ldots, L_{(N-1)+}$ (cf. Fig. 3) the existence of skein relations is deduced from the properties of the partition function. The inner product of (17) with $\langle\chi|$ yields

$$\alpha\,\langle\chi|\psi\rangle + \alpha_1\,\langle\chi|\psi_1\rangle + \cdots + \alpha_N\,\langle\chi|\psi_N\rangle = 0, \tag{18}$$

and implies a higher-dimensional skein relation of order $d = N-1$ of the form

$$\begin{aligned}&\alpha_- Z_{G,R}(M, L_-) + \alpha_0 Z_{G,R}(M, L_0) + \alpha_+ Z_{G,R}(M, L_+)\\&\quad + \cdots + \alpha_{(N-1)+} Z_{G,R}(M, L_{(N-1)+}) = 0.\end{aligned} \tag{19}$$

The monodromy operation B defined for each Lie group of the various Cartan types can be used to generate the vectors ψ_i from ψ [37,38]:

$$B^j|\psi_i\rangle = |\psi_{i+j}\rangle, \qquad B^j|\psi\rangle = |\psi_j\rangle \quad (j \in \mathbb{Z}). \tag{20}$$

Following Witten the characteristic equation of the $N \times N$ half-twist monodromy matrix B can be written as

$$\prod_{i=1}^{N}(B - \lambda_i) = 0. \tag{21}$$

The eigenvalues λ_i can be deduced from the monodromy properties of the four-point correlation function for the primary fields in representation R of Wess–Zumino conformal field theory on S^2 [37,38]:

$$\lambda_i = (-1)^{N+i} \exp[\mathrm{i}\pi(2h_R - h_{E_i})], \tag{22}$$

where h_R and h_{E_i} are the conformal weights of the primary conformal field transforming as R and E_i, respectively. Using the dependency relation (17) and the characteristic equation (21) acting on $|\psi\rangle$ with no twists, yields the skein relation coefficients of (19):

$$\begin{aligned}
\alpha_- &= (-1)^N \prod_{i=1}^{N} \lambda_i, \\
\alpha_0 &= (-1)^{N-1} \left(\prod_{i=1}^{N} \lambda_i\right) \left(\sum_{i=1}^{N} \lambda_i^{-1}\right), \\
&\vdots \\
\alpha_{(N-2)+} &= -\sum_{i=1}^{N} \lambda_i, \\
\alpha_{(N-1)+} &= 1.
\end{aligned} \tag{23}$$

In order to reach agreement with the notation used in knot literature employing standard framing, the Wilson lines in the manifold X_j must be adjusted by j-fold Dehn twists [17]. This imposes a subsequent multiplication of the coefficients α_{j+} by $\exp[-j\pi \mathrm{i} h_R]$ ($j = 0, \ldots, N-1$), respectively.

According to Govindarajan [29] for links obtained as closure of braids made from two strands carrying the same representation R of $G = SU(n)$, the invariants may be evaluated explicitly without the necessity of solving recursive skein relations using

$$Z_{SU(n),R}(M, L_{j+}) = \sum_{i=1}^{N} (\dim_q E_i)\, \lambda_i^j \tag{24}$$

with the q-dimension of the IR E_i.

Let C_v be the quadratic Casimir operator of the adjoint representation related to the dual Coxeter number of the selected Lie group. Then the usual q variable substitution

$$q = \exp\left[\frac{2\pi \mathrm{i}}{k + C_v}\right] \tag{25}$$

imposes simultaneously the deformation parameter of quantum groups related to primary fields of Wess–Zumino conformal fields.

Employing now the fundamental representation R of the semi-simple Lie groups of Cartan's classification A_{n-1}, B_n, C_n or D_n, the theory will engender the A,B,C,D-IRF-model invariants. In terms of Young tableaux the decomposition relation reads

Table 1
The ABCD-IRF-model invariants

Gauge group	IRF-model invariant	
A_{n-1}	α_-	q^n
	α_0	$q^{n/2-1/2} - q^{n/2+1/2}$
	α_+	-1
B_n	α_-	$-q^{4n}$
	α_0	$q^{2n} - q^{3n-1/2} + q^{3n+1/2}$
	α_+	$q^{n-1/2} - q^{n+1/2} + q^{2n}$
	α_{2+}	-1
C_n	α_-	q^{4n+2}
	α_0	$q^{2n+1} - q^{3n+2} + q^{3n+1}$
	α_+	$q^n - q^{n+1} + q^{2n+1}$
	α_{2+}	-1
D_n	α_-	$-q^{4n-2}$
	α_0	$q^{2n-1} - q^{3n-1} + q^{3n-2}$
	α_+	$q^{n-1} - q^n + q^{2n-1}$
	α_{2+}	-1

$$\begin{aligned} \square \otimes \square &= \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\square \qquad &&\text{for } A_{n-1},\\ \square \otimes \square &= \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\square \oplus \phi \quad &&\text{for } B_n, C_n, D_n, \end{aligned} \tag{26}$$

where $E_i = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, $\square\square$, ϕ are antisymmetric, symmetric and scalar representations of G, respectively. The conformal weights h_{E_i} are calculated (cf. [38,24]) allowing to obtain the skein relation coefficients (23) straightforwardly as listed for the convenience of the reader in Table 1.

4. Composite A,B,C,D-IRF-model invariants

Recalling the fusion procedure to construct higher spin representations by combining $m = 2s$ strings it is significant to generalize the theory proposed using higher-dimensional representations of the selected gauge group.

According to Gepner [39] solvable IRF lattice models are in one-to-one correspondence with a pair of a rational conformal field theory (RCFT) and a field in it. As a consequence of that one can form for each such pair an associated link invariant [40] performed here by means of topological CS theory. The projection operator of the monodromy matrix B_k at the face k on a primary field a is defined by

$$P^a = \prod_{i=1, i\neq a}^{N} \frac{B_k - \lambda_i}{\lambda_a - \lambda_i}. \tag{27}$$

The corresponding IRF model may then be introduced via its Boltzmann weights described in the usual operator form (cf. [40]):

$$X_k(u) = \sum_{a=1}^{N} P_k^a f^a(u) \tag{28}$$

with the functions

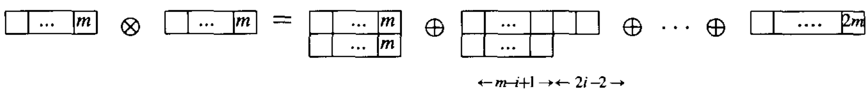
$$f^a(u) = \prod_{i=1}^{a-1} \sin(\zeta_i + u) \prod_{i=1}^{N-1} \sin(\zeta_i - u), \tag{29}$$

where

$$\zeta_i = \tfrac{1}{2}\pi [h_{E_{i+1}} - h_{E_i}]. \tag{30}$$

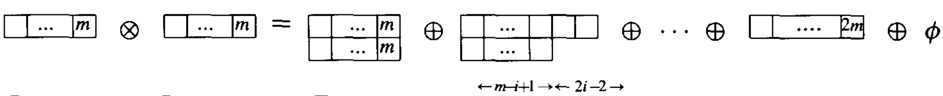
Here u is the spectral parameter which labels the family of models. Making use of the properties of RCFT and the IRF models [40], it follows that the invariants so constructed always obey the Markov properties, and thus are true link invariants.

As consequence of that property composite braid group relations may be considered using Witten's CS theory with a spin $s = \frac{1}{2}(N-1)$ representation of the gauge group G. This representation is given by Young tableaux containing $m = 2s$ boxes in a row and the generalized decomposition relation is given diagrammatically by



R_s R_s E_1 E_i E_{m+1}

for A_{n-1},



R_s R_s E_1 E_i E_{m+1}

for B_n, C_n, D_n.

(31)

Claiming more mathematical exactness especially for the cases of the special orthogonal Lie groups, it is necessary to express the decomposition of the representations in terms of Dynkin's notation. The Dynkin coefficients a_i of the concerned IR R_s, E_i and ϕ are $\{m, 0, \ldots, 0\}$, $\{2i-2, m-i+1, 0, \ldots, 0\}$ and $\{0, \ldots, 0\}$, respectively.

The conformal weights h of the representation Γ are related to the second Casimir operator C_Γ of this representation as [22]

$$h_\Gamma = \frac{1}{k + C_v}\, C_\Gamma. \tag{32}$$

The highest weight Λ of the IR under consideration is usually expressed (see, e.g. [41]) as a sum of the weights l_i of the fundamental representation

$$\Lambda = \sum_i l_i \lambda_i, \tag{33}$$

with the labeling

$$\begin{aligned}
A_{n-1}:\ & l_k = l_{n+1} + \sum_{i=k}^{n} a_i \quad (1 \le k \le n), \quad l_{n+1} = -\frac{1}{n+1}\sum_{i=1}^{k} i a_i, \\
B_n:\quad & l_k = \frac{a_n}{2} + \sum_{i=k}^{n-1} a_i, \\
C_n:\quad & l_k = \sum_{i=k}^{n} a_i, \\
D_n:\quad & l_k = \frac{a_{n-1} - a_n}{2} + \sum_{i=k}^{n-2} a_i.
\end{aligned} \tag{34}$$

Considering a convenient normalization of the fundamental weights λ_i [38] the quadratic Casimir may be evaluated for arbitrary IR for any semi-simple Lie group leading in particular to the results

$$\begin{aligned}
A_{n-1}:\ & C_\Gamma = \frac{1}{2}\sum_{i=1}^{n+1} l_i (l_i - 2i), \\
B_n:\quad & C_\Gamma = \frac{1}{2}\sum_{i=1}^{n} l_i (l_i + 2n - 2i + 1), \\
C_n:\quad & C_\Gamma = \frac{1}{2}\sum_{i=1}^{n} l_i (l_i + 2n - 2i + 2), \\
D_n:\quad & C_\Gamma = \frac{1}{2}\sum_{i=1}^{n} l_i (l_i + 2n - 2i).
\end{aligned} \tag{35}$$

Accordingly the conformal weights of the representations corresponding to the concerned decomposition (31) are straightforwardly obtained: ($i = 1, \ldots, 2s+1 = 1, \ldots, m+1$)

Table 2
Eigenvalues of the monodromy matrices corresponding to the composite ABCD-IRF-model invariants

Gauge group	Eigenvalues of the composite ABCD-IRF-model invariants ($i=1,\ldots,m+1$)	
A_{n-1}	λ_{E_i}	$(-1)^{i+m+1}q^{-(1/2)i(i-1)+m(m+n)/2n}$
B_n	λ_{E_i}	$(-1)^{i+m+1}q^{-i^2/2+i/2+m/2}$
	λ_ϕ	$q^{(m/2)[m+2n-1]}$
C_n	λ_{E_i}	$(-1)^{i+m+1}q^{-i^2/2+i/2+m/2}$
	λ_ϕ	$-q^{(m/2)[m+2n]}$
D_n	λ_{E_i}	$(-1)^{i+m+1}q^{-i^2/2+i/2+m/2}$
	λ_ϕ	$q^{(m/2)[m+2n-2]}$

$$
\begin{aligned}
A_{n-1}: \quad h_{R_s} &= \frac{1}{k+C_v}\frac{m(m+n)(n-1)}{2n},\\
h_{E_i} &= \frac{1}{k+C_v}\frac{i^2n-in+m(n-2)(m+n-2)}{n},\\
B_n: \quad h_{R_s} &= \frac{1}{k+C_v}\frac{m}{2}[m+2n-1],\\
h_{E_i} &= \frac{1}{k+C_v}[i^2-i+m^2+2mn-2m],\\
C_n: \quad h_{R_s} &= \frac{1}{k+C_v}\frac{m}{2}[m+2n],\\
h_{E_i} &= \frac{1}{k+C_v}[i^2-i+m^2+2mn-m],\\
D_n: \quad h_{R_s} &= \frac{1}{k+C_v}\frac{m}{2}[m+2n-2],\\
h_{E_i} &= \frac{1}{k+C_v}[i^2-i+m^2+2mn-3m].
\end{aligned}
\tag{36}
$$

Representing the main result of this work, the defining skein relations of composite A,B,C,D-IRF-model invariants (19) may now be determined from the eigenvalues of the monodromy matrices λ_i (22) listed in Table 2. Recalling that $h_\phi = 0$, the eigenvalues related to the scalar representation follow from the conformal weights of the spin-s representation (cf. (22)).

Recalling the standard framing adjustment by j-fold Dehn twists of the Wilson lines with the factor $\exp[-j\pi i h_{R_s}]$ ($j = 0, \ldots, N-1$) mentioned below (23) the skein relation coefficients $\alpha_-, \alpha_0, \ldots, \alpha_{(N-1)+}$ are obtained explicitly.

The corresponding braid operators of $\mathcal{B}_k^{[s]}$

$$G_i^{\pm} = \lim_{u \to \pm\infty} \frac{X_i(u)}{\varphi(u)} \tag{37}$$

with

$$\varphi(u) \equiv f^N(u) = \prod_{i=1}^{N-1} \sin(\zeta_i + u) \tag{38}$$

(cf. (28)) obey the Markov properties [40] and the resulting composite A,B,C,D-IRF-model invariants are consequently well defined.

The composite A_{n-1} model invariants (${}^m A_{n-1}$ invariants) agree exactly with the original defining relations of the generalized N-vertex-model invariants (5) of Wadati et al. [8,11] for $N = 2s + 1 = m + 1$ (cf. (1)).

For example in the case of $m = 1$ ($N = 2$) the eigenvalues related to the ${}^1 A_{n-1}$ invariants are

$$\lambda_1 = \lambda_{E_1} = -q^{(1+n)/2n}, \qquad \lambda_2 = \lambda_{E_2} = q^{(1-n)/2n}, \tag{39}$$

yielding the HOMFLY polynomial (8), whereas for $m \geq 2$ ($N \geq 3$) the well-known skein relations of higher order (cf. (9)) may be derived.

The link polynomials for Lie groups of Cartan's classification B_n, C_n and D_n are new. Taking into account that now applies

$$N = m + 2 \tag{40}$$

for example in the cases of B_n with $m = 2$ (spin 1) the eigenvalues given in Table 2 are explicitly

$$\lambda_1 = \lambda_{E_1} = q^{1/2},\ \lambda_2 = \lambda_{E_2} = -q^{-1/2},\ \lambda_3 = \lambda_{E_3} = q^{-5/2},\ \lambda_4 = \lambda_{\phi} = q^{2n+1} \tag{41}$$

leading to the skein relation of the new ${}^2 B_n$ link invariant ($\omega = q^{n-1}$)

$$\begin{aligned} &-\omega^6 q^{-23/2} L_- + \omega^{-4}[q^{-2n-19/2} + q^{-6} - q^{-8} + q^{-9}] L_0 \\ &\quad + \omega^{-2}[-q^{-2n-4} + q^{-2n-6} - q^{-2n-7} + q^{-5/2} - q^{-7/2} + q^{-11/2}] L_+ \\ &\quad + \omega^{-2}[-q^{-2n-2} - q^{-5/2} + q^{-7/2} - q^{-11/2}] L_{2+} + L_{3+} = 0. \end{aligned} \tag{42}$$

The generalized A,B,C,D-IRF-model invariants depending on the spin $s = \frac{1}{2}m$ provide a possibility to classify the previously found two-variable vertex- and the IRF-model invariants indicated in Table 3.

Furthermore the conformity of some important link polynomials derived from integrable models of statistical mechanics may be reviewed. The agreement of certain link polynomials derived from integrable vertex- or IRF-models may be understood recalling the Wu–Kadanoff–Wegener transformation [42,43] which indicates the specific relationship of Boltzmann weights of both types of models.

Table 3
Classification model of link invariants derived from solvable models of statistical mechanics

Cartan type	A_{n-1}		B_n	C_n	D_n
Lie group	$SU(n=2)$	$SU(n)$	$SO(2n+1)$	$Sp(2n)$	$SO(2n)$
$m=1$	JONES polynomial A_1 IRF-model invariant 2-vertex model invariant	HOMFLY polynomial A_{n-1} IRF-model invariant Composite 2-vertex model invariant	B,C,D IRF-MODEL invariant		
$m \in \mathbb{N}$	$N=(m+1)$vertex model invariants Composite A_1 IRF-model invariant	2-var. $N=(m+1)$ vert. model invariants Composite A_{n-1} IRF model invariant	Comp. B,C,D IRF-model invariants		

The generalized surgery formula of Witten [17] for the diffeomorphism K of the boundary of the corresponding manifold M,

$$Z_G(\tilde{M}, R_i) = \sum_j K_i^j \, Z_G(M, R_j), \tag{43}$$

allows to calculate the invariants for a new three-manifold $\tilde{M}$ containing a Wilson line in the R_i representation from those obtained in the manifold M with a Wilson line of associated representation R_j. In the case of $\tilde{M} = S^3$ and $M = S^2 \times S^1$, K becomes the known modular group diffeomorphism S and surgery gives rise to the possibility to evaluate $Z_G(S^3)$ from $Z_G(S^2 \times S^1, R_j)$. $Z_G(S^3, L(\bigcup_i R_i))$ may be calculated as for example

$$Z_G(S^3, L(R_i, R_j)) = \sum_k S_i^k Z_G(S^2 \times S^1, R_k, R_j) = S_{i,j} \tag{44}$$

yielding Witten's invariant for the unknotted rings

$$\begin{aligned} Z_{G,R}(M, \circ\circ) &= Z_{G,R}(M, \circ)\frac{S_{R,0}}{S_{0,0}} \\ &= Z_{G,R}(M,\circ)\left[\frac{q^{-2C_R} - q^{2C_R}}{\sum_i (-1)^{i+m+1}(q^{-C_{E_i}} - q^{C_{E_i}})} - \prod_i (-1)^{i+m+1}\right]. \end{aligned} \tag{45}$$

Furthermore the change of the orientation of some link components in the case of real representations R results in multiplying another factor of q related to the writhe $w(k)$ of the knot

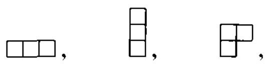
$$Z_{G,R}(M, L) = q^{C_R w(K)} Z_{G,R}(M, L'). \tag{46}$$

Concluding from the analogy with the non-composite case [21], the knot polynomials for closed three braids can be derived directly without using recursive skein relations.

Certainly, the theory proposed for composite link invariants may be extended to the exceptional Lie groups classified by G_2, F_4, $E_{6,7,8}$, or applied to arbitrary decompositions into irreducible representations. The latter case complies with the generalization of the composite string representation with projectors chosen as eigenvectors of the full twist Δ_i^2 (10). The eigenvalues are calculated from the Dynkin coefficients in the form

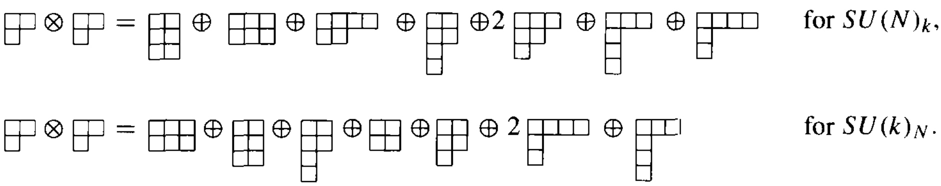
$$a_\lambda = q^{\frac{1}{2}m(m-1)-\frac{1}{2}\sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty} a_i\right)\left(\sum_{i=j}^{\infty} a_i - 2j\right)}. \tag{47}$$

Following Wadati et al. [8] for example in the case of ${}^3A_1$ there exist $N = m + 1 = 4$ projectors: the symmetrizer, the antisymmetrizer, two projectors with mixed symmetry corresponding to the Young tableaux



respectively. Consequently the generalized invariants so obtained coincide with the extended Akutsu–Wadati invariants $\alpha_\omega^{[\lambda]}$ given in (11).

It is interesting to observe that the duality relation between $SU(N)_k$ and $SU(k)_N$ Wess–Zumino–Witten models [44,45] is reflected in the link invariants derived by topological means as intended here. This may be verified examining, e.g., the respective decomposition of the representation  :



(48)

Since the permutation of the monodromy eigenvalues λ_i, associated with irreducible representations E_i, does not lead to new results of the skein relation coefficients (cf. (23)), the same link invariants will be obtained for dual decompositions. These results are a consequence of the duality properties of the spaces of conformal blocks of $SU(N)_k$ and $SU(k)_N$ correlation functions and their associated braid matrices [45].

Observe that the extended version of Witten's Chern–Simons theory implies the possibility to derive composite invariants of non-standard representations of Li and Ge [15,16]. These non-standard representations are sequences of new solutions of the spectral parameter-independent Yang–Baxter equation, wherein the coefficients of the Kauffman diagrams depend on the possible labelings [13]. According to Li and Ge the eigenvalues of the corresponding monodromy matrices are given by

$$\begin{aligned} \lambda_1 &= q, \quad \lambda_2 = -q^{-1} && \text{for } A_{n-1}, B_n, C_n, D_n, \\ \lambda_3 &= q^{-\mu+1} && \text{for } B_n, \\ \lambda_3 &= -\delta_1\, q^{-\mu-\delta_1} && \text{for } C_n, \\ \lambda_3 &= \delta_1\, q^{-\mu+\delta_1} && \text{for } D_n. \end{aligned} \tag{49}$$

The parameters μ and δ_1 may be calculated from the possible sets of labelings of the diagrammatic Yang–Baxter equation [15,16], providing the possibility to construct composite

Table 4
The invariants of the non-standard representation

Gauge group	Non-standard invariant	
A_{n-1}	α_-	$q^{-2\mu}$
	α_0	$q^{1-\mu} - q^{-1-\mu}$
	α_+	-1
B_n	α_-	$-q^{-4\mu+4}$
	α_0	$q^{-2\mu+2} - q^{-3\mu+4} + q^{-3\mu+2}$
	α_+	$q^{-\mu+2} - q^{-\mu} + q^{-2\mu+2}$
	α_{2+}	-1
C_n	α_-	$\delta_1 q^{-4\mu-4\delta_1}$
	α_0	$\delta_1(q^{-3\mu-3\delta_1-1} - q^{-3\mu-3\delta_1+1} - \delta_1 q^{-2\mu-2\delta_1})$
	α_+	$q^{-\mu-\delta_1+1} - q^{-\mu-\delta_1-1} - \delta_1 q^{-2\mu-2\delta_1}$
	α_{2+}	-1
D_n	α_-	$-\delta_1 q^{-4\mu+4\delta_1}$
	α_0	$\delta_1(q^{-3\mu+3\delta_1-1} - q^{-3\mu+3\delta_1+1} + \delta_1 q^{-2\mu+2\delta_1})$
	α_+	$q^{-\mu+\delta_1+1} - q^{-\mu+\delta_1-1} - \delta_1 q^{-2\mu+2\delta_1}$
	α_{2+}	-1

non-standard invariants by similar means. Note that there exists an interesting agreement of the non-standard invariants obtained before and listed in Table 4 (cf. [24]) with some composite invariants. For example in the case of B_3 for the special set of labelings with $\mu = 3$ the invariants turn out to be equivalent to the composite ${}^2A_1$ invariant (or three-vertex Akutsu–Wadati invariant) for $q = \exp[-2\pi\mathrm{i}/(k + C_v)]$.

Finally one might examine the asymmetrical decomposition relations

$$R \otimes R' = \bigoplus_{i=1}^{N} E_i. \tag{50}$$

In this case the eigenvalues of the monodromy matrices of RCFT are related to the conformal weights as

$$\lambda_i = (-1)^{N+i} \exp[\mathrm{i}\pi(h_R + h_{R'} - h_{E_i})], \tag{51}$$

and the invariants may be obtained in the same way as indicated above.

5. Conclusion

The present report supplies an intrinsically 3D definition of composite A,B,C,D-IRF-model invariants in the framework of Witten's topological field theory based on the pure Chern–Simons action. The knowledge of the expectation values of Wilson operators over links with arbitrary representation of arbitrary semi-simple Lie groups constitutes a complete solution for the non-abelian Chern–Simons theory in three dimensions. The construction presented here, while encompassing the known link invariants derived from integrable models of statistical mechanics, provides a generalization of these along with a summary of the framework for their construction. The composite invariants obtained and related to the fusion procedure of factorized scattering matrices provide a possibility of classification for the previously found link invariants showing once more the universality of Witten's pioneering work. Using the conventions of this approach the HOMFLY polynomial and the two-variable Akutsu–Wadati vertex invariants correspond to the ${}^1A_{n-1}$ and ${}^mA_{n-1}$ composite model invariants, respectively. The generalized invariants related to Lie groups of Cartan's classification B_n, C_n or D_n, however, do not have a "classical" precedence.

Such new invariants with skein relations of higher order are indeed needed in the problem of classifying links since two topological links may certainly have identical classifying invariants. The most famous example is the Birman pair of two different knots having the same Jones polynomial [46]. Furthermore as shown recently by Govindarajan et al. [47] the chirality of the two knots 9_{42} and 10_{71} is not detected by any of the well-known polynomials, namely Jones, HOMFLY and Kauffman. However, the composite ${}^mA_{n-1}$ invariants are indeed sensitive to the chirality of these knots for $m \geq 3$, providing a systematic classification possibility for the invariants. Higher spin polynomials being progressively more powerful emphasizes the importance of the introduction of composite A,B,C,D-IRF-model invariants. This appears to be significant when recalling that the problem of classifying link invariants is the same as that of classifying conformal field theory since the link invariants described here arise from RCFT.

Moreover the universality of Witten's topological field theory implies a far reaching consequence for the notion of quantum groups. These are related to certain q-deformations $U_q\mathcal{G}$ of universal enveloping algebras of classical Lie algebras, where q is restricted to be the complex root of unity given by (25). Since quantum groups are intimately connected with solvable models of statistical mechanics it was possible to establish a close relationship between invariants of closed three-manifolds and the quantum enveloping algebras [48–51]. For example the Jones polynomial is known to be connected with the quantum envelopping algebra of the Lie algebra $sl_2(\mathrm{C})$. In the same way Witten succeeded in deriving the corresponding structure coefficients starting only from the general covariance of 3D Chern–Simons theory with gauge group $SU(2)$ [19], it is essentially possible to generalize Witten's approach to arbitrary classical Lie groups as, e.g., in the case of link invariants. This implies a partial explanation of the existence of the quantum groups $SL_q(n)$, $GL_q(n)$, $O_q(n)$ and $Sp_q(n)$ emerging as a unifying structure between integrable 2D field theories, 3D CS gauge theory and link invariants. Hereby the quantum symmetry appears as a hidden symmetry of Witten's partition function.

Further generalizations may be reviewed when exploring higher-dimensional irreducible representations of the basic classical Lie superalgebras $SU(n|n')$, $OSp(2n+1|2n')$, $OSp(2|2n-2)$, $OSp(2n|2n')$, $G(3)$ or $F(4)$. The results are of particular interest since it is essentially possible to introduce the Boltzmann weights of supersymmetric IRF models from the monodromy eigenvalues in analogy with the non-graded case (cf. (28)).

An alternative 3D variational approach of Cotta-Ramusino et al. [52] allows to compute the skein coefficients in the case of $G = SU(n)$ in a large coupling approximation $k \to \infty$. The method may eventually be extended for arbitrary classical Lie groups.

Moreover there is a fundamental relationship between Jones' knot invariants and Vassiliev's knot invariants derived from important concepts of classical topology. The corresponding connection to more sophisticated invariants still seems to be elusive.

Finally the reflection of duality properties of link invariants determined from $SU(n)_k$ and $SU(k)_n$ Wess–Zumino–Witten models deserve a further examination.

In conclusion the derivation of composite link invariants in the framework of Chern–Simons topological gauge theory may suggest new promising developments for knot theory and the interpretation of quantum groups providing a further small step in the understanding of the close relationship between topological field theory, integrable models in statistical mechanics and the concept of link invariants.

Acknowledgements

It is a pleasure to thank A. Holz, T.R. Govindarajan, L.H. Kauffman, P. Ramadevi and V.G. Turaev for several useful discussions. Furthermore I have benefitted from helpful advice via e-mail from J. Fuchs, D. Gepner, H. Osborn and E. Witten. This work was partially supported by the Landesgraduiertenförderung of the Saarland.

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